

# SMALL DATA SCATTERING FOR THE NONLINEAR SCHRÖDINGER EQUATION ON PRODUCT SPACES

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**ABSTRACT.** We consider the cubic nonlinear Schrödinger equation, posed on  $\mathbb{R}^n \times M$ , where  $M$  is a compact Riemannian manifold and  $n \geq 2$ . We prove that under a suitable smallness in Sobolev spaces condition on the data there exists a unique global solution which scatters to a free solution for large times.

## 1. INTRODUCTION

The Cauchy problem for the nonlinear Schrödinger equation (NLS), posed on a compact Riemannian manifold attracted a considerable attention (see in particular [1, 2]). In all these works the global existence is based on the combination of (low regularity) well-posedness and conservation laws. In such a situation there is few control on the global dynamics and in particular there is no reason to believe that the solution of the nonlinear problem is close to the solution of the linear problem for large times, even for small data. In other words scattering is not expected.

On the other hand the Cauchy problem for the nonlinear Schrödinger equation, posed on the Euclidean space  $\mathbb{R}^n$  is better understood (see for instance [4, 6, 9]). In particular for small data one expects that the nonlinear evolution is close to the linear one, at least for small data and sufficiently small (near zero) nonlinearity, see for instance [4]. An important tool in the proof of such type of results are the *global in time* Strichartz estimates for the linear Schrödinger evolution on  $\mathbb{R}^n$ . Such type of global in time estimates are false when the problem is posed on a compact manifold.

In view of the previous discussion a natural problem is to consider the NLS on  $\mathbb{R}^n \times M$ , where  $M$  is a compact Riemannian manifold. This is the purpose of this paper. We will show that the global in time dispersive nature of the  $\mathbb{R}^n$  part is still sufficient to get small data scattering results similarly to the Euclidean case. Our view point is to see the problem as a NLS type equation for functions on  $\mathbb{R}^n$  with values in Sobolev spaces on  $M$  (instead of  $\mathbb{C}$  for the "usual" NLS). We should however admit that our approach, as presented here, is not working for problems such as the wave equation on product spaces.

In order to emphasize the main ideas of the paper and to avoid technicalities we will restrict our attention to the cubic nonlinear interaction, even if our approach can be extended to other nonlinearities. Consider thus the Cauchy problem

$$(1.1) \quad i\partial_t u + \Delta_{x,y} u = \pm |u|^2 u, \quad u(0, x, y) = f(x, y),$$

with  $(t, x, y) \in \mathbb{R}_t \times \mathbb{R}_x^n \times M_y^k$ , where  $M_y^k$  is a compact Riemannian manifold of dimension  $k \geq 1$  and  $\Delta_{x,y} = \Delta_x + \Delta_y$  with  $\Delta_y$  the Laplace-Beltrami operator on  $M_y^k$  and  $\Delta_x = \sum_{j=1}^n \partial_{x_j}^2$  is the Laplace operator associated to the flat metric on  $\mathbb{R}^n$ .

We are interested here in the scattering of global solutions to (1.1), under suitable smallness assumptions on the initial data. In order to state our first result, we introduce a non isotropic Sobolev space. Namely, we denote by  $\mathcal{H}_{x,y}^{\theta,\rho}$  and the completions of  $C_0^\infty(\mathbb{R}_x^n \times M_y^k)$  with respect to the following norm

$$\|f\|_{\mathcal{H}_{x,y}^{\theta,\rho}} = \sum_{|\alpha| \leq \theta} \|\partial_x^\alpha (1 - \Delta_y)^{\rho/2} f\|_{L^2(\mathbb{R}_x^n \times M_y^k)},$$

where for  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ ,  $\partial_x^\alpha \equiv \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$  and  $|\alpha| \equiv \alpha_1 + \dots + \alpha_n$ . Here is our first result.

**Theorem 1.1.** *Let  $n \geq 2$  be even. Then for every  $\epsilon > 0$  there exists  $\delta = \delta(\epsilon) > 0$  such that the Cauchy problem (1.1) has an unique global solution*

$$u(t, x, y) \in L_t^\infty \mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon} \cap X_\epsilon$$

where

$$(1.2) \quad \|u\|_{X_\epsilon} = \sum_{s=0}^{\frac{n-2}{2}} \sum_{|\alpha|=s} \|\partial_x^\alpha (1 - \Delta_y)^{\frac{1}{2}(\frac{k}{2} + \epsilon)} u\|_{L_t^4 L_x^{\frac{2n}{1+2s}} L_y^2}$$

for any initial data  $f(x, y) \in \mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon}$  such that  $\|f\|_{\mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon}} < \delta$ . Moreover there exist  $f_0^\pm \in \mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon}$  such that

$$(1.3) \quad \lim_{t \rightarrow \pm\infty} \|e^{it\Delta_{x,y}} f_0^\pm - u(t, x, y)\|_{\mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon}} = 0.$$

*Remark 1.1.* The evenness of  $n$  in Theorem 1.1 is needed since in this case we are able to estimate the cubic nonlinearity in the space  $X_\epsilon$  by using the usual Leibniz rule. The case  $n$  odd is treated in Theorem 1.2 below.

It is interesting to compare Theorem 1.2 in the case  $n = k = 2$  with the recent result [5]. In [5], the problem (1.1) on  $\mathbb{R}^2 \times \mathbb{T}^2$  is considered ( $\mathbb{T}^2$  is the flat 2d torus) and the global well-posedness for small data in the classical Sobolev spaces  $H^1(\mathbb{R}^2 \times \mathbb{T}^2)$  is proved. The scattering to free solution is not obtained in [5], the globalization argument being based on conservation laws together with Tataru's critical spaces theory. Therefore our result says that in the context of the analysis in [5] if in addition one supposes the smallness of the  $\mathcal{H}^{0,1+\epsilon}$  norm then one has scattering. Note that the  $\mathcal{H}^{0,1+\epsilon}$  norm is slightly stronger than the  $H^1(\mathbb{R}^2 \times \mathbb{T}^2)$  only with respect the  $y$  variables. Since in our analysis we do not use any dispersive effect in  $y$  it would be interesting to further understand the interplay between our argument in the case  $\mathbb{R}^2 \times \mathbb{T}^2$  and the corresponding analysis in  $H^1(\mathbb{R}^2 \times \mathbb{T}^2)$  in [5]. It is also worth noticing that our argument here is only restricted to the small data cases while the analysis of [5] also applies to the large data problem, if we consider sub-cubic defocusing nonlinear interactions.

We next turn to the odd dimensional case. In this case  $(n-2)/2$  is not an integer and a direct application of the proof of Theorem 1.1 would require some non trivial non isotropic Littlewood-Paley theory. We decided not to pursue this. Instead, we apply a simple argument which reduces the case of  $n \geq 3$  odd to the case of  $n$  even. For  $n \geq 3$ , we define  $\mathcal{H}_{x,(x_n,y)}^{\theta,\rho}$  to be the completion of  $C_0^\infty(\mathbb{R}_x^n \times M_y^k)$  with respect

to the following norm

$$\|f\|_{\mathcal{H}_{\bar{x},(x_n,y)}^{\theta,\rho}} = \sum_{|\alpha| \leq \theta} \|\partial_{\bar{x}}^\alpha (1 - \partial_{x_n}^2 - \Delta_y)^{\rho/2} f\|_{L^2(\mathbb{R}_x^n \times M_y)},$$

where  $\bar{x} = (x_1, \dots, x_{n-1})$  and for  $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{N}^{n-1}$ ,  $\partial_{\bar{x}}^\alpha \equiv \partial_{x_1}^{\alpha_1} \dots \partial_{x_{n-1}}^{\alpha_{n-1}}$ . Here is our result concerning the odd dimensions  $n \geq 3$ .

**Theorem 1.2.** *Let  $n \geq 3$  be odd. Then for every  $\epsilon > 0$  there exists  $\delta = \delta(\epsilon) > 0$  such that the Cauchy problem (1.1) has an unique global solution*

$$u(t, x, y) \in L_t^\infty \mathcal{H}_{\bar{x},(x_n,y)}^{\frac{n-3}{2}, \frac{k+1}{2} + \epsilon} \cap X_\epsilon$$

where

$$(1.4) \quad \|u\|_{X_\epsilon} = \sum_{s=0}^{\frac{n-3}{2}} \sum_{|\alpha|=s} \|\partial_{\bar{x}}^\alpha (1 - \partial_{x_n}^2 - \Delta_y)^{\frac{k+1}{4} + \epsilon} u\|_{L_t^4 L_x^{\frac{2(n-1)}{1+2s}} L_{(x_n,y)}^2}$$

for any initial data  $f(x, y) \in \mathcal{H}_{\bar{x},(x_n,y)}^{\frac{n-3}{2}, \frac{k+1}{2} + \epsilon}$  such that  $\|f\|_{\mathcal{H}_{\bar{x},(x_n,y)}^{\frac{n-3}{2}, \frac{k+1}{2} + \epsilon}} < \delta$ . Moreover

there exist  $f_0^\pm \in \mathcal{H}_{\bar{x},(x_n,y)}^{\frac{n-3}{2}, \frac{k+1}{2} + \epsilon}$  such that

$$(1.5) \quad \lim_{t \rightarrow \pm\infty} \|e^{it\Delta_{x,y}} f_0^\pm - u(t, x, y)\|_{\mathcal{H}_{\bar{x},(x_n,y)}^{\frac{n-3}{2}, \frac{k+1}{2} + \epsilon}} = 0.$$

An analogue of Theorem 1.2 can not hold for  $n = 1$ . In this case we expect a suitable Banach space valued version of the modified scattering result of Ozawa [7]. Such a result will give an insight into the global small data dynamics of the cubic NLS on  $\mathbb{R} \times \mathbb{T}$ , established in [8].

In the case  $n = 1$  one can obtain, by invoking a Banach space valued version of the small data theory of the quintic NLS on  $\mathbb{R}$ , an analogue of Theorem 1.2 if the cubic non linearity is replaced by the quintic one, namely  $\pm|u|^2u$  replaced by  $\pm|u|^4u$ .

The remaining part of the paper is organized as follows. In the next section, we establish a basic Strichartz inequality. This inequality only uses the dispersive effect in the  $x$  variables but have the advantage to be global in time. Next, we prove Theorem 1.1. The final section is devoted to the proof of Theorem 1.2.

## 2. A STRICHARTZ TYPE INEQUALITY

In this section, we establish our basic tool which is a Strichartz type estimate for  $e^{it\Delta_{x,y}}$ . Here is the precise statement.

**Proposition 2.1.** *For every  $n \geq 1$  and for every compact Riemannian manifold  $M_y^k$  the following estimate holds:*

$$(2.1) \quad \|e^{it\Delta_{x,y}} f\|_{L_t^p L_x^q L_y^2} + \left\| \int_0^t e^{i(t-\tau)\Delta_{x,y}} F(\tau, x, y) d\tau \right\|_{L_t^p L_x^q L_y^2} \leq C(\|f\|_{L_{x,y}^2} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'} L_y^2}),$$

where  $C = C(p, q, \tilde{p}, \tilde{q}) > 0$  and

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad \frac{2}{\tilde{p}} + \frac{n}{\tilde{q}} = \frac{n}{2}$$

$2 \leq p, \tilde{p} \leq \infty$  for  $n > 2$ ,  $2 < p, \tilde{p} \leq \infty$  for  $n = 2$  and  $4 \leq p, \tilde{p} \leq \infty$  for  $n = 1$ .

*Proof.* Let us recall the usual Strichartz estimates for the free propagators  $e^{it(\Delta_x+m)}$  on  $\mathbb{R}_x^n$  with  $m \in \mathbb{R}$ :

$$(2.2) \quad \sup_{m \in \mathbb{R}} (\|e^{it(\Delta_x+m)} h\|_{L_t^p L_x^q} + \|\int_0^t e^{i(t-\tau)(\Delta_x+m)} H(\tau, x) d\tau\|_{L_t^p L_x^q}) \\ \leq C(\|h\|_{L_x^2} + \|H\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}})$$

under the same assumptions on  $p, \tilde{p}, q, \tilde{q}$ , with  $C = C(p, \tilde{p}, q, \tilde{q}) > 0$  that does not depend on  $m$ . Recall that the usual Strichartz estimate concerns the propagator  $e^{it\Delta_x}$ . On the other hand in (2.2) we are allowed to get uniform bounds with respect to  $m \in \mathbb{R}$  since  $e^{it(\Delta_x+m)} = e^{itm} e^{it\Delta_x}$  and moreover the Strichartz norm are not affected by the remodulation factor  $e^{itm}$ . Next we introduce

$$u(t, x, y) = e^{it\Delta_{x,y}} f + \int_0^t e^{i(t-\tau)\Delta_{x,y}} F(\tau, x, y) d\tau$$

and notice that

$$i\partial_t u + \Delta_x u + \Delta_y u = F, \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}_x^n \times M_y$$

with

$$u(0, x, y) = f(x, y).$$

Let us decompose

$$u(t, x, y), f(x, y) \text{ and } F(t, x, y)$$

with respect to the orthonormal basis  $\{\varphi_j(y)\}$  of  $L^2(M_y)$  given by the eigenfunctions of  $-\Delta_y$  (i.e.  $-\Delta_y \varphi_j = \lambda_j \varphi_j$ )

$$(2.3) \quad u(t, x, y) = \sum_j u_j(t, x) \varphi_j(y)$$

$$(2.4) \quad F(t, x, y) = \sum_j F_j(t, x) \varphi_j(y)$$

$$f(x, y) = \sum_j f_j(x) \varphi_j(y)$$

and notice that  $u_j(t, x)$ ,  $F_j(t, x)$  and  $f_j(x)$  are related by the following Cauchy problems:

$$(2.5) \quad i\partial_t u_j + \Delta_x u_j - \lambda_j u_j = F_j, \quad (t, x) \in \mathbb{R}_t \times \mathbb{R}_x^n$$

with

$$u_j(0, x) = f_j(x).$$

Applying (2.2) in the context of (2.5) gives

$$\|u_j(t, x)\|_{L_t^p L_x^q} \leq C\|f_j\|_{L^2} + C\|F_j(t, x)\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}$$

and hence summing in  $j$  the squares we get

$$\|u_j(t, x)\|_{l_j^2 L_t^p L_x^q} \leq C\|f\|_{L_{x,y}^2} + C\|F_j(t, x)\|_{l_j^2 L_t^{\tilde{p}'} L_x^{\tilde{q}'}}.$$

On the other hand

$$\max\{\tilde{p}', \tilde{q}'\} \leq 2 \leq \min\{p, q\}$$

and therefore by the Minkowski inequality we get

$$\|u_j(t, x)\|_{L_t^p L_x^q l_j^2} \leq C\|f\|_{L_{x,y}^2} + C\|F_j(t, x)\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'} l_j^2}.$$

Combining (2.3) and (2.4) with the Plancharel identity gives

$$\|u\|_{L_t^p L_x^q L_y^2} \leq C\|f\|_{L_{x,y}^2} + C\|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'} L_y^2}.$$

Finally, we apply the last inequality first with  $f = 0$  and then  $F = 0$  to achieve the bound (2.1). This completes the proof of Proposition 2.1.  $\square$

### 3. PROOF OF THEOREM 1.1

We recall a suitable version of Strichartz estimates for the classical propagator  $e^{it(\Delta_x+m)}$  on  $\mathbb{R}_x^n$  with  $n$  an even integer and  $m \in \mathbb{R}$ . Notice that we use that  $n$  is even in order to have that  $(n-2)/2$  is an integer which allows us to give a meaning to the derivation operators up to  $(n-2)/2$  that appears on the r.h.s. of (3.1) (see also below remark 3.1).

**Proposition 3.1.** *Let  $n \geq 2$  be even,  $\alpha \in \mathbb{N}^n$  such that  $0 \leq |\alpha| \leq \frac{n-2}{2}$ , then*

$$(3.1) \quad \sup_{m \in \mathbb{R}} \left( \|\partial_x^\alpha e^{it(\Delta_x+m)} h\|_{L_t^p L_x^q} + \|\partial_x^\alpha \left( \int_0^t e^{i(t-\tau)(\Delta_x+m)} H(\tau, x) d\tau \right)\|_{L_t^p L_x^q} \right) \\ \leq C \left( \|h\|_{H_x^{\frac{n-2}{2}}} + \sum_{|\beta|=\frac{n-2}{2}} \|\partial_x^\beta H\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \right),$$

where  $C = C(p, q, \tilde{p}, \tilde{q}) > 0$  does not depend on  $m \in \mathbb{R}$ ,

$$\frac{2}{p} + \frac{n}{q} = 1 + |\alpha|, \quad \frac{2}{\tilde{p}} + \frac{n}{\tilde{q}} = \frac{n}{2}, \quad 2 < p, \tilde{p} \leq \infty.$$

*Proof.* Observe that, under our restriction on  $p$ , we have  $1 < q < \infty$ . We have the Sobolev embedding

$$\dot{W}^{\frac{n-2}{2}-|\alpha|, q_1}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \frac{n}{q_1} - \frac{n}{q} = \frac{n-2}{2} - |\alpha| \implies \frac{2}{p} + \frac{n}{q_1} = \frac{n}{2}.$$

Therefore the left hand-side of (3.1) is bounded by

$$\sup_{m \in \mathbb{R}} \sum_{|\beta|=\frac{n-2}{2}} \left( \|\partial_x^\beta e^{it(\Delta_x+m)} h\|_{L_t^p L_x^{q_1}} + \|\partial_x^\beta \left( \int_0^t e^{i(t-\tau)(\Delta_x+m)} H(\tau, x) d\tau \right)\|_{L_t^p L_x^{q_1}} \right).$$

Thus the estimate (3.1) for a fixed  $m$  follows the usual Strichartz estimates thanks to the relation  $\frac{2}{p} + \frac{n}{q_1} = \frac{n}{2}$ . The uniformity of the estimate with respect to  $m \in \mathbb{R}$  can be deduced as in (2.2).  $\square$

*Remark 3.1.* Notice that in the case  $n$  odd an estimate similar to (3.1) is satisfied provided that the local operator  $\partial_x^\alpha$  is replaced by  $(1 - \Delta_x)^{|\alpha|/2}$ . However for our purpose it will be relevant to work with  $\partial_x^\alpha$ , in view of the possibility to apply for this operator the usual Leibniz rule for the derivation of a product.

The next result will be fundamental in the sequel.

**Proposition 3.2.** *Let  $n \geq 2$  be even,  $\alpha \in \mathbb{N}^n$  such that  $0 \leq |\alpha| \leq \frac{n-2}{2}$  and  $r \geq 0$ , then we have*

$$(3.2) \quad \begin{aligned} & \|\partial_x^\alpha (1 - \Delta_y)^{r/2} e^{it\Delta_{x,y}} f\|_{L_t^p L_x^q L_y^2} \\ & + \|\partial_x^\alpha (1 - \Delta_y)^{r/2} \left( \int_0^t e^{i(t-\tau)\Delta_{x,y}} F(\tau) d\tau \right)\|_{L_t^p L_x^q L_y^2} \\ & \leq C \left( \|f\|_{\mathcal{H}_{x,y}^{\frac{n-2}{2}, r}} + \sum_{|\beta| = \frac{n-2}{2}} \|\partial_x^\beta (1 - \Delta_y)^{r/2} F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'} L_y^2} \right), \end{aligned}$$

where  $C = C(p, \tilde{p}, q, \tilde{q}) > 0$ ,

$$\frac{2}{p} + \frac{n}{q} = 1 + |\alpha|, \quad \frac{2}{\tilde{p}} + \frac{n}{\tilde{q}} = \frac{n}{2}, \quad 2 < p, \tilde{p} \leq \infty.$$

*Proof.* It is sufficient to consider the case  $r = 0$ . The general case follows simply by a derivation of the equation with respect to  $y$  variables. The proof of (3.2) for  $r = 0$  is similar to the proof of Proposition 2.1 by using Proposition 3.1 instead of (2.2).  $\square$

In the sequel we shall work with the following norm:

$$(3.3) \quad \|u\|_{X_\epsilon} = \sum_{s=0}^{\frac{n-2}{2}} \sum_{|\alpha|=s} \|\partial_x^\alpha (1 - \Delta_y)^{\frac{1}{2}(\frac{k}{2} + \epsilon)} u\|_{L_t^4 L_x^{\frac{2n}{1+2s}} L_y^2}.$$

**Proposition 3.3.** *Assume  $n$  is even and  $\epsilon > 0$ . Then there exists  $C = C(n, \epsilon) > 0$  such that for every  $\beta \in \mathbb{N}^d$  satisfying  $|\beta| = \frac{n-2}{2}$ , every  $u_1, u_2, u_3 \in X_\epsilon$ ,*

$$\|\partial_x^\beta (1 - \Delta_y)^{\frac{1}{2}(\frac{k}{2} + \epsilon)} (u_1 u_2 u_3)\|_{L_t^{\frac{4}{3}} L_x^{\frac{2n}{n+1}} L_y^2} \leq C \|u_1\|_{X_\epsilon} \|u_2\|_{X_\epsilon} \|u_3\|_{X_\epsilon}.$$

*Proof.* Notice that  $\mathcal{H}_y^{\frac{k}{2} + \epsilon}$  is an algebra and hence

$$(3.4) \quad \|(1 - \Delta_y)^{\frac{1}{2}(\frac{k}{2} + \epsilon)} (f_1 f_2 f_3)\|_{L_y^2} \leq C \prod_{j=1}^3 \|(1 - \Delta_y)^{\frac{1}{2}(\frac{k}{2} + \epsilon)} f_j\|_{L_y^2}.$$

Moreover, by the Leibnitz formula

$$(3.5) \quad \partial_x^\beta (g_1 g_2 g_3) = \sum_{|\beta_1| + |\beta_2| + |\beta_3| = \frac{n-2}{2}} c_{\beta_1 \beta_2 \beta_3} (\partial_x^{\beta_1} g_1 \partial_x^{\beta_2} g_2 \partial_x^{\beta_3} g_3)$$

for a suitable choice of the coefficient  $c_{\beta_1 \beta_2 \beta_3}$ . By combining (3.4), (3.5) with the Minkowski inequality it is sufficient to prove

$$\left\| \prod_{j=1}^3 \|(1 - \Delta_y)^{\frac{1}{2}(\frac{k}{2} + \epsilon)} \partial_x^{\beta_j} u_j\|_{L_y^2} \right\|_{L_t^{\frac{4}{3}} L_x^{\frac{2n}{n+1}}} \leq C \|u_1\|_{X_\epsilon} \|u_2\|_{X_\epsilon} \|u_3\|_{X_\epsilon}$$

with  $|\beta_1| + |\beta_2| + |\beta_3| = \frac{n-2}{2}$ . Using the relation

$$\frac{n+1}{2n} = \frac{1+2|\beta_1|}{2n} + \frac{1+2|\beta_2|}{2n} + \frac{1+2|\beta_3|}{2n}$$

and the Hölder inequality, applied with respect to  $(t, x)$ , we get

$$\begin{aligned} & \left\| \prod_{j=1}^3 \|(1 - \Delta_y)^{\frac{1}{2}(\frac{k}{2} + \epsilon)} \partial_x^{\beta_j} u_j\|_{L_y^2} \right\|_{L_t^{\frac{4}{3}} L_x^{\frac{2n}{n+1}}} \leq \\ & \prod_{j=1}^3 \left\| (1 - \Delta_y)^{\frac{1}{2}(\frac{k}{2} + \epsilon)} \partial_x^{\beta_j} u_j \right\|_{L_t^{\frac{4}{3}} L_x^{\frac{2n}{2|\beta_j|+1}} L_y^2} \leq \|u_1\|_{X_\epsilon} \|u_2\|_{X_\epsilon} \|u_3\|_{X_\epsilon}. \end{aligned}$$

This completes the proof of Proposition 3.3.  $\square$

**Proof of Theorem 1.1** The problem (1.1) can be rewritten as the integral equation

$$(3.6) \quad u(t) = e^{it\Delta_{x,y}} f \pm \int_0^t e^{i(t-s)\Delta_{x,y}} (|u(s)|^2 u(s)) ds \equiv T_f(u).$$

The proof of (1.3) is standard once it is proved the existence of a global solution  $u(t, x, y)$  belonging to

$$Y_\epsilon = L_t^\infty \mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon} \cap X_\epsilon$$

and hence will be omitted (for more details on this fact see [3]). By a fixed point argument it is sufficient to prove the following

*Claim:*

$$\begin{aligned} & \forall \epsilon \in (0, \infty) \exists \delta = \delta(\epsilon) > 0 \text{ and } R = R(\epsilon) > 0 \text{ s.t. } T_f(Y_{\epsilon,R}) \subset Y_{\epsilon,R} \\ & \text{and } T_f \text{ is a contraction on } Y_{\epsilon,R} \forall f \text{ s.t. } \|f\|_{\mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon}} < \delta, \end{aligned}$$

where  $Y_{\epsilon,R} = \{u \in Y_\epsilon \mid \|u\|_{Y_\epsilon} < R\}$ .

By combining (3.2) with Proposition 3.3 we get:

$$\begin{aligned} \|T_f u\|_{Y_\epsilon} & \leq C(\|f\|_{\mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon}} + \sum_{|\beta| = \frac{n-2}{2}} \|\partial_x^\beta (1 - \Delta_y)^{\frac{1}{2}(\frac{k}{2} + \epsilon)} (u|u|^2)\|_{L_t^{\frac{4}{3}} L_x^{\frac{2n}{n+1}} L_y^2}) \\ & \leq C(\|f\|_{\mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon}} + \|u\|_{X_\epsilon}^3) \\ & \leq C(\|f\|_{\mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon}} + \|u\|_{Y_\epsilon}^3). \end{aligned}$$

By a standard continuity argument the previous estimate gives the existence of  $\delta > 0$  and  $R(\delta) > 0$  such that

$$T_f(Y_{\epsilon,R(\delta)}) \subset Y_{\epsilon,R(\delta)}$$

provided that  $\|f\|_{\mathcal{H}_{x,y}^{\frac{n-2}{2}, \frac{k}{2} + \epsilon}} < \delta$ . Moreover  $\lim_{\delta \rightarrow 0} R(\delta) = 0$ . Next we shall check that  $T_f$  is a contraction on  $Y_{\epsilon,R(\delta)}$  provided that  $\delta$ , and hence  $R(\delta)$ , are small. By using (3.2) we get

$$\|T_f(v) - T_f(w)\|_{Y_\epsilon} \leq C \sum_{|\beta| = \frac{n-2}{2}} \|\partial_x^\beta (1 - \Delta_y)^{\frac{1}{2}(\frac{k}{2} + \epsilon)} (v|v|^2 - w|w|^2)\|_{L_t^{\frac{4}{3}} L_x^{\frac{2n}{n+1}} L_y^2}$$

that in conjunction with the following identity

$$v^2 \bar{v} - w^2 \bar{w} = (v - w)(v + w) \bar{w} + v^2 (\bar{v} - \bar{w})$$

and with Proposition 3.3 gives

$$\|T_f(v) - T_f(w)\|_{Y_\epsilon} \leq C\|v - w\|_{Y_\epsilon}(\|v\|_{Y_\epsilon} + \|w\|_{Y_\epsilon})^2 \leq C\|v - w\|_{Y_\epsilon}(R(\delta))^2.$$

Hence  $T_f$  is contraction on  $Y_{\epsilon, R(\delta)}$  in case  $R(\delta)$  is small enough.

#### 4. PROOF OF THEOREM 1.2

Notice that if we split

$$\mathbb{R}_x^n \times M_y^k = \mathbb{R}_{\bar{x}}^{n-1} \times (\mathbb{R}_{x_n} \times M_y^k)$$

then we are reduced to the situation of Theorem 1.1 since  $n - 1$  is an even number. However we are not allowed to apply directly Theorem 1.1 since the manifold  $\mathbb{R}_{x_n} \times M_y^k$  is not compact (despite to the assumption of Theorem 1.1). To overcome this difficulty we shall prove the following version of Proposition 3.2.

**Proposition 4.1.** *Let  $n \geq 3$  be odd,  $\alpha \in \mathbb{N}^n$  such that  $0 \leq |\alpha| \leq \frac{n-3}{2}$  and  $r \geq 0$ , then*

$$(4.1) \quad \begin{aligned} & \|\partial_{\bar{x}}^\alpha (1 - \partial_{x_n}^2 - \Delta_y)^{r/2} e^{it\Delta_{x,y}} f\|_{L_t^p L_{\bar{x}}^q L_{(x_n,y)}^2} \\ & + \|\partial_{\bar{x}}^\alpha (1 - \partial_{x_n}^2 - \Delta_y)^{r/2} \left( \int_0^t e^{i(t-\tau)\Delta_{x,y}} F(\tau) d\tau \right)\|_{L_t^p L_{\bar{x}}^q L_{(x_n,y)}^2} \\ & \leq C \left( \|f\|_{\mathcal{H}_{\bar{x},(x_n,y)}^{\frac{n-3}{2},r}} + \sum_{|\beta|=\frac{n-3}{2}} \|\partial_{\bar{x}}^\beta (1 - \partial_{x_n}^2 - \Delta_y)^{r/2} F\|_{L_t^{\tilde{p}'} L_{\bar{x}}^{\tilde{q}'} L_{(x_n,y)}^2} \right), \end{aligned}$$

where

$$\frac{2}{p} + \frac{n-1}{q} = 1 + |\alpha|, \quad \frac{2}{\tilde{p}} + \frac{n-1}{\tilde{q}} = \frac{n-1}{2}, \quad 2 < p, \tilde{p} \leq \infty.$$

*Proof.* It is sufficient to consider the case  $r = 0$ . The general case follows by a derivation with respect to the  $y$  variables. Let

$$u(t, \bar{x}, x_n, y) = e^{it\Delta_{x,y}} f + \int_0^t e^{i(t-\tau)\Delta_{x,y}} F(\tau) d\tau.$$

Then

$$i\partial_t u + \Delta_{\bar{x}} + \partial_{x_n}^2 u + \Delta_y u = F$$

with

$$u(0, x, y) = f(x, y).$$

Next we introduce the partial Fourier transform of  $u, f, F$  with respect to the  $x_n$  variable

$$\hat{u}(t, \bar{x}, \xi_n, y), \hat{f}(\bar{x}, \xi_n, y) \text{ and } \hat{F}(t, \bar{x}, \xi_n, y),$$

which satisfy

$$i\partial_t \hat{u} + \Delta_{\bar{x}} \hat{u} - \xi_n^2 \hat{u} + \Delta_y \hat{u} = \hat{F}, \quad (t, \bar{x}, y) \in \mathbb{R}_t \times \mathbb{R}_{\bar{x}}^{n-1} \times M_y^k$$

with

$$\hat{u}(0, \bar{x}, \xi_n, y) = \hat{f}(\bar{x}, \xi_n, y).$$

Next, we decompose

$$\hat{u}(t, \bar{x}, \xi_n, y), \hat{f}(\bar{x}, \xi_n, y) \text{ and } \hat{F}(t, \bar{x}, \xi_n, y)$$



with respect to the orthonormal basis  $\{\varphi_j\}$  of  $L^2(M_y)$  given by the eigenfunctions of  $-\Delta_y$  (i.e.  $-\Delta_y \varphi_j = \lambda_j \varphi_j$ .) Then we have

$$\begin{aligned}\hat{u}(t, \bar{x}, \xi_n, y) &= \sum_j \hat{u}_j(t, \bar{x}, \xi_n) \varphi_j(y) \\ \hat{F}(t, \bar{x}, \xi_n, y) &= \sum_j \hat{F}_j(t, \bar{x}, \xi_n) \varphi_j(y) \\ \hat{f}(\bar{x}, \xi_n, y) &= \sum_j \hat{f}_j(\bar{x}, \xi_n) \varphi_j(y).\end{aligned}$$

Moreover  $\hat{u}_j(t, \bar{x}, \xi_n)$ ,  $\hat{f}_j(\bar{x}, \xi_n)$  and  $\hat{F}_j(t, \bar{x}, \xi_n)$  are related by the following Cauchy problems

$$(4.2) \quad \mathbf{i} \partial_t \hat{u}_j + \Delta_{\bar{x}} \hat{u}_j - \xi_n^2 \hat{u}_j - \lambda_j \hat{u}_j = \hat{F}_j, \quad (t, \bar{x}, y) \in \mathbb{R}_t \times \mathbb{R}_{\bar{x}}^{n-1} \times M_y^k$$

with

$$\hat{u}(0, \bar{x}, \xi_n) = \hat{f}(\bar{x}, \xi_n).$$

Using Proposition 3.1 in the context of (4.2) gives

$$\|\partial_{\bar{x}}^s \hat{u}_j(t, \bar{x}, \xi_n)\|_{L_t^p L_{\bar{x}}^q} \leq C \|\hat{f}_j(\bar{x}, \xi_n)\|_{H_{\bar{x}}^{\frac{n-3}{2}}} + C \sum_{|\beta|=\frac{n-3}{2}} \|\partial_{\bar{x}}^\beta \hat{F}_j(t, \bar{x}, \xi_n)\|_{L_t^{\tilde{p}'} L_{\bar{x}}^{\tilde{q}'}}$$

where  $C = C(p, \tilde{p}, q, \tilde{q}) > 0$  is constant uniform with respect to  $j$  and  $\xi_n$  and  $p, \tilde{p}, q, \tilde{q}$  are as in the assumptions. In particular we get

$$\|\partial_{\bar{x}}^s \hat{u}_j(t, \bar{x}, \xi_n)\|_{L_{\xi_n}^2 L_j^2 L_t^p L_{\bar{x}}^q} \leq C \|f\|_{\mathcal{H}_{\bar{x}, (x_n, y)}^{\frac{n-3}{2}, r}} + C \sum_{|\beta|=\frac{n-3}{2}} \|\partial_{\bar{x}}^\beta \hat{F}_j(t, \bar{x}, \xi_n)\|_{L_{\xi_n}^2 L_j^2 L_t^{\tilde{p}'} L_{\bar{x}}^{\tilde{q}'}}.$$

Again, we use that

$$\max\{\tilde{p}', \tilde{q}'\} \leq 2 \leq \min\{p, q\}$$

and therefore the Minkowski inequality gives

$$\|\partial_{\bar{x}}^s \hat{u}_j(t, \bar{x}, \xi_n)\|_{L_t^p L_{\bar{x}}^q L_{\xi_n}^2 L_j^2} \leq C \|f\|_{\mathcal{H}_{\bar{x}, (x_n, y)}^{\frac{n-3}{2}, r}} + C \sum_{|\beta|=\frac{n-3}{2}} \|\partial_{\bar{x}}^\beta \hat{F}_j(t, \bar{x}, \xi_n)\|_{L_t^{\tilde{p}'} L_{\bar{x}}^{\tilde{q}'} L_{\xi_n}^2 L_j^2}.$$

Now, the proof can be concluded by the Plancharel identity (with respect to  $x_n$  and  $y$ ) as we did in Proposition 2.1.  $\square$

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1 and involves the following version of Proposition 3.3.

**Proposition 4.2.** *Let  $n \geq 3$  be odd and  $r > \frac{k+1}{2}$ . Then we have the following trilinear estimate*

$$\sum_{|\beta|=\frac{n-3}{2}} \|\partial_{\bar{x}}^\beta (1 - \partial_{x_n}^2 - \Delta_y)^{r/2} (u_1 u_2 u_3)\|_{L_t^{\frac{4}{3}} L_{\bar{x}}^{\frac{2(n-1)}{n}} L_{(x_n, y)}^2} \leq C \|u_1\|_X \|u_2\|_X \|u_3\|_X$$

where

$$\|u\|_X = \sum_{s=0}^{\frac{n-3}{2}} \sum_{|\alpha|=s} \|\partial_{\bar{x}}^\alpha (1 - \partial_{x_n}^2 - \Delta_y)^{r/2} u\|_{L_t^4 L_{\bar{x}}^{\frac{2(n-1)}{1+2s}} L_{(x_n, y)}^2}.$$

*Proof.* See the proof of Proposition 3.3.  $\square$

**Proof of Theorem 1.2.** It is similar to the proof of Theorem 1.1 provided that Proposition 4.1 and 4.2 are used instead of Proposition 3.2 and 3.3.

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